Q1

Monday, August 29, 2011
10:20 AM

Consider $X \sim \text{Bernoulli}(p)$

$$H(X) = -\sum_{x} p_X(x) \log_2 p_X(x)$$

$$= -(1-p)\log(1-p) - p\log p$$

Now take $\lim_{p \to 0}$ or $\lim_{p\to 1}$, we have $H(X) \to 0$

$$\left(\lim_{x\to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0 \right)$$

L'Hôpital's rule

So, if p is close to 1 or 0, the entropy will be almost 0. But we still need 1 bit to send X. So, the expected length will be very close to $H(X) + 1$.

Remark: The cases when $p=0$ and $p=1$ can not be used here. These cases correspond to degenerated X. ($X \equiv 0$ and $X \equiv 1$, respectively)

Because, in each case, X is a constant, we don't have to transmit any bit to communicate the value of X. Alternatively, we may think of this as sending the empty string (ε) to the receiver.

So, $\mathbb{E}[\ell(X)] = \mathbb{E}[0] = 0$ which is the same as $H(X)$.

$H(X)$.

Q2

Tuesday, September 20, 2011

11:05 AM

AWGN → BSC

(a) First, note that to have $\hat{X} = a$, we need $Y \geq 0$.

Now, $Y = X + N$. Hence, we need $N \geq 0 - X = -X$.

Therefore, $P[\hat{X}=a \mid X = \alpha] = P[N \geq -X \mid X = \alpha]$

$= P[N \geq -\alpha \mid X = \alpha]$

$X \perp\!\!\!\perp N \rightarrow$ $= P[N \geq -\alpha] = P[N > -\alpha]$

N is a cont. RV.

Recall that for
$N \sim \mathcal{N}(m, \sigma^2)$,
$P[N > n] = P[N \geq n] = Q\left(\frac{n-m}{\sigma}\right)$
$= 1 - \Phi\left(\frac{n-m}{\sigma}\right)$

$$\left.\begin{aligned} &= Q\left(\frac{-\alpha-0}{\sigma_N}\right) = Q\left(\frac{-\alpha}{\sigma_N}\right). \\ &= 1 - \Phi\left(\frac{-\alpha}{\sigma_N}\right) = \Phi\left(\frac{\alpha}{\sigma_N}\right) \end{aligned}\right\} \star$$

Similarly, to have $\hat{X} = -a$, we need $Y < 0$.

From $Y = X + N$, we need $N < 0 - X = -X$.

Therefore, $P[\hat{X} = -a \mid X = \alpha] = P[N < -X \mid X = \alpha]$

$= P[N < -\alpha \mid X = \alpha]$

$X \perp\!\!\!\perp N \rightarrow$ $= P[N < -\alpha] = P[N \leq -\alpha]$

N is a cont. RV

$$\left.\begin{aligned} &= \Phi\left(\frac{-\alpha-0}{\sigma_N}\right) = \Phi\left(\frac{-\alpha}{\sigma_N}\right) \\ &= 1 - Q\left(\frac{-\alpha}{\sigma_N}\right) = Q\left(\frac{\alpha}{\sigma_N}\right) \end{aligned}\right\} \star\star$$

Alternatively, we can use the fact that $\hat{X}$ can only take two values, "a" and "$-a$". Hence,

$$P[\hat{X} = -a \mid X = \alpha] = 1 - P[\hat{X} = a \mid X = \alpha].$$

(i) $P[\hat{X} = -a \mid X = a] \overset{\star\star}{=} Q\left(\frac{a}{\sigma_N}\right)$

(ii) $P[\hat{X} = a \mid X = a] \overset{\star}{=} \Phi\left(\frac{a}{\sigma_N}\right)$

(ii) $P[\hat{X} = a \mid X = a] \overset{*}{=} \Phi\left(\frac{a}{\sigma_N}\right)$

(iii) $P[\hat{X} = -a \mid X = -a] \overset{**}{=} \Phi\left(\frac{a}{\sigma_N}\right)$

(iv) $P[\hat{X} = a \mid X = -a] \overset{*}{=} Q\left(\frac{a}{\sigma_N}\right)$

In this part, $a = 5$ and $\sigma_N = 3 \Rightarrow \frac{a}{\sigma_N} = \frac{5}{3} \approx 1.67$

From [Y & G, Table 3.1], $\Phi(1.67) \approx 0.9525$.

So, $Q(1.67) \approx 1 - 0.9525 = 0.0475$.

Final answers:

(i) 0.0475 (ii) 0.9525 (iii) 0.9525 (iv) 0.0475

Remark: If you use the exact value $\left(\frac{5}{3}\right)$ with MATLAB, then the answers will be 0.0478 and 0.9522.

(v) $P[\hat{X} \neq X] = P[\hat{X} = -a \mid X = a] P[X = a] + P[\hat{X} = a \mid X = -a] P[X = -a]$

$$= Q\left(\frac{a}{\sigma_N}\right) P[X = a] + Q\left(\frac{a}{\sigma_N}\right) P[X = -a]$$

$$= Q\left(\frac{a}{\sigma_N}\right) \left(\underbrace{P[X = a] + P[X = -a]}_{1}\right) = Q\left(\frac{a}{\sigma_N}\right) \approx 0.0475$$

Remark: we refer to the probability in part (v) as the error probability which is the same as the bit error rate when X is binary.

Note also that it may written as $Q\left(\sqrt{SNR}\right)$ where $SNR = \frac{a^2}{\sigma_N^2}$ = signal (power) to noise (power) ratio.

(b) p = crossover probability = our answers in part (a.i) and (a.iv)

$$= Q\left(\frac{a}{\sigma_N}\right) = Q\left(\frac{5}{3}\right) \approx Q(1.67) \approx 0.0475.$$

(c) No. As $\sigma_N \to \infty$, $\frac{a}{\sigma_N}$ will decrease to 0 and $Q(0) = \frac{1}{2}$.

(d) As shown in part (b), $p = Q\left(\frac{a}{\sigma_N}\right)$.

Q3

Tuesday, October 09, 2012
8:37 AM

(a) We want to compare $P[S=0|Z=1]$ and $P[S=1|Z=1]$.

By Bayes' theorem,

$$P[S=0|Z=1] = \frac{P[Z=1|S=0]\,P[S=0]}{P[Z=1]} = \frac{p\times(1-p_1)}{P[Z=1]} \quad \text{and}$$

$$P[S=1|Z=1] = \frac{P[Z=1|S=1]\,P[S=1]}{P[Z=1]} = \frac{(1-p)\times p_1}{P[Z=1]}$$

Note that both conditional probabilities above have $P[Z=1]$ in their denominators and hence to determine which one is larger, we can ignore the $P[Z=1]$ part.

The table below compares the values of $p(1-p_1)$ and $(1-p)p_1$ for the four given senarios.

	p	p_1	$p(1-p_1)$		$(1-p)p_1$
(i)	0.3	0.1	0.27	$>$	0.07
(ii)	0.3	0.5	0.15	$<$	0.35
(iii)	0.3	0.9	0.03	$<$	0.63
(iv)	0.7	0.5	0.35	$>$	0.15

At the receiver, to determine what was transmitted when $Z=1$ is observed, we compare $P[S=s|Z=1]$ for all possibilities of S. (For us, S can be 0 or 1.) To minimize the probability of decoding error, the (MAP) decoder set

$\hat{S}$ to be the value of s that maximize $P[S=s|Z=1]$.

From the table above, we can determine the most likely transmitted bit

(i) 0 (ii) 1 (iii) 1 (iv) 0

(b) Again, we compare

$$P[S=0|Z=0] = \frac{P[Z=0|S=0]\,P[S=0]}{P[Z=0]} = \frac{(1-p)(1-p_1)}{P[Z=0]} \quad \text{and}$$

$$P[S=1|Z=0] = \frac{P[Z=0|S=1]\,P[S=1]}{P[Z=0]} = \frac{p \times p_1}{P[Z=0]}.$$

p	p_1	$(1-p)(1-p_1)$		$p \times p_1$
0.3	0.1	0.63	>	0.03
0.3	0.5	0.35	>	0.15
0.3	0.9	0.07	<	0.27
0.7	0.5	0.15	<	0.35

(i) 0 (ii) 0 (iii) 1 (iv) 1

Remark : When we combine your answer from parts (a) and (b), we can see that the MAP detectors are different for different values of p and p_1

p	p_1	$\hat{S}_{MAP}(Z)$	
0.3	0.1	0	← always guess 0
0.3	0.5	Z	← always follow the output of BSC
0.3	0.9	1	← always guess 1
0.7	0.5	$\bar{Z}$	← always guess the opposite of the output of BSC

Q4

Tuesday, October 09, 2012
8:21 AM

(a)

The transmitted bit is repeated 5 times.
The decoder uses majority vote and hence will decode incorrectly if there are ≥ 3 bits (of the five bits) that get switched over BSC.

Therefore, $P(\mathcal{E}) = \underbrace{\sum_{k=3}^{5} \binom{5}{k} p^k (1-p)^{5-k}}$ $\approx$ **0.3174** (substitute $p = 0.4$)

Let Z be the number of bits that get switched over BSC.
Then Z is a binomial (n, p) RV.

n: 5

p: crossover probability

$$P(\mathcal{E}) = \sum_{k=\lceil \frac{n}{2} \rceil}^{n} P[Z=k] = 1 - \sum_{k=\lfloor \frac{n}{2} \rfloor}^{0} P[Z=k] = 1 - F_Z\left(\lfloor \tfrac{n}{2} \rfloor\right)$$

$= 1 - \text{binocdf}\left(\lfloor \frac{n}{2} \rfloor, n, p\right)$ in MATLAB

(b) Let $\underline{Z}$ be the five observed bits at the receiver.

(i) $$P[S=0 \mid \underline{Z} = 01001] = \frac{P[\underline{Z} = 01001 \mid S=0]\, P[S=0]}{P[\underline{Z} = 01001]}$$

$$= \frac{p^2(1-p)^3 \times p_0}{P[\underline{Z} = 01001]}$$

$P[S=0] = 0.45$

conditioned on $S=0$,
the repetition code would be 00000
and therefore, $\underline{Z} = 01001$ means
the BSC flips exactly two bits.

By the total probability theorem,

$$P[\underline{Z} = 01001] = P[\underline{Z} = 01001 \mid S=0]\, P[S=0]$$

$$+ P[\underline{Z} = 01001 \mid S = 1]\, P[S = 1]$$

$$= p^2(1-p)^3 p_0 + p^3(1-p)^2 \underbrace{p_1}$$

$P[S = 1] = 0.55$

$$\approx 0.0282$$

Therefore, $P[S=0 \mid \underline{Z} = 01001] \approx 0.5510$

(ii) $P[S=1 \mid \underline{Z} = 01001] = 1 - P[S=0 \mid \underline{Z} = 01001] = 0.4490$

(iii) $P[S=0 \mid \underline{Z} = 01001] > P[S=1 \mid \underline{Z} = 01001]$.

Therefore, it is more likely that $S=0$ was transmitted.

(c) (i) $$P[S=0 \mid \underline{Z} = 01001] = \frac{p^2(1-p)^3 \times p_0}{p^2(1-p)^3 p_0 + p^3(1-p)^2 p_1} = \frac{1}{1 + \frac{p}{1-p}\frac{p_1}{p_0}}$$

$$= \frac{1}{1 + \frac{2}{3} \times \frac{1-p_0}{p_0}} = \frac{3p_0}{p_0 + 2}$$

(ii) $$P[S=1 \mid \underline{Z} = 01001] = 1 - \frac{3p_0}{p_0+2} = \frac{2 - 2p_0}{p_0 + 2}$$

(iii) $P[S=0 \mid \underline{Z} = 01001]$ is more likely iff

$$\frac{3p_0}{p_0+2} > \frac{2-2p_0}{p_0+2}$$

$$p_0 > \frac{2}{5} = 0.4$$

$P[S=1 \mid \underline{Z} = 01001]$ is more likely iff $p_0 < 0.4$

When $p_0 = 0.4$, the conditional probabilities are the same and the two cases are equally likely.

Majority voting would always guess 0 because the #0s in 01001 is greater than the #1s. This would agree with our answer here when $p_0 > 0.4$.

Q5

Tuesday, October 09, 2012
9:50 AM

In this question, there are four possible codewords.

Let $\underline{X}$ be the transmitted codeword.

$\underline{Y}$ be the observed bits at the receiver.

Observe that $P[\underline{Y} = \underline{y} \mid \underline{X} = \underline{x}] = p^d (1-p)^{n-d}$

where d = Hamming distance between $\underline{x}$ and $\underline{y}$

(the key is to realize that the Hamming distance gives the number of bits that get switched over the BSC.)

and n = length of the codeword.

For example,

one bit switched

$$P[\underline{Y} = 01001 \mid \underline{X} = 01000] = p^1 (1-p)^{5-1}.$$

(b) We find $\underline{x}$ that would maximize $P[X = \underline{x} \mid Y = 01001]$.

Now,
$$P[X = \underline{x} \mid Y = \underline{y}] = \frac{P[Y = \underline{y} \mid X = \underline{x}]\, P[X = \underline{x}]}{P[Y = \underline{y}]}$$

This does not depend on $\underline{x}$

This is assumed to be uniform and therefore it is a constant.

So, to compare $P[X = \underline{x} \mid Y = \underline{y}]$ for different $\underline{x}$,

we can simply compare $P[Y = \underline{y} \mid X = \underline{x}] = p^d (1-p)^{n-d}$.

Note that for $p < 0.5$, $p^d (1-p)^{n-d}$ is a strictly decreasing function of d. To see this note that for $d_1 > d_2$,

$$p^{d_1} (1-p)^{n-d_1} = \underbrace{p \times p \times \cdots \times p}_{d_1 \text{ times}} \times \underbrace{(1-p) \times (1-p) \times \cdots \times (1-p)}_{n-d_1 \text{ times}}$$

$$< \underbrace{p \times p \times \cdots p}_{d_2 \text{ times}} \underbrace{(1-p)(1-p)}_{d_1 - d_2} \times \underbrace{(1-p) \times (1-p) \times \cdots \times (1-p)}_{n-d_1 \text{ times}}.$$

$$< \underbrace{p \times p \times \cdots p}_{d_2 \text{ times}} \; \underbrace{(1-p)(1-p)}_{d_1 - d_2 \text{ times}} \times \underbrace{(1-p) \times (1-p) \times \cdots \times (1-p)}_{n - d_1 \text{ times}} .$$

We can then conclude that the $\underline{x}$ that maximize $P[X = \underline{x} \mid Y = \underline{y}]$ is the same as the $\underline{x}$ that minimizes the Hamming distance d between $\underline{x}$ and $\underline{y}$. This is the same as the minimum distance decoder.

(a) In part (b), we have shown that the most likely transmitted codeword is the one that minimize the Hamming distance from the received. So, we find the distances in the table below

$\underline{x}$	$d(\underline{x}, \underline{y})$, $\underline{y} = 01001$	
00000	2	
01000	1	← minimum
10001	2	
11111	3	

From the table, we see that the most likely transmitted codeword is 01000.

(c)

$d_{min} = 1$

$\underline{x}$	00000	01000	10001	11111
00000		1	2	5
01000			3	4
10001				3
11111				

(d) From part (b), recall that we need to compare

$$P[\underline{X} = \underline{x} \mid \underline{Y} = \underline{y}] = \frac{P[\underline{Y} = \underline{y} \mid \underline{X} = \underline{x}] \, P[\underline{X} = \underline{x}]}{P[\underline{Y} = \underline{y}]}$$

$$= \frac{p^d (1-p)^{n-d} \, P[\underline{X} = \underline{x}]}{P[\underline{Y} = \underline{y}]}$$

The main difference here is that we need to take into account $P[\underline{X} = \underline{x}]$.

$\underline{x}$	$d(\underline{x},\underline{y})$	$P[\underline{X} = \underline{x}]$	$p^d(1-p)^{n-d} P[\underline{X} = \underline{x}]$
00000	2	0.1	0.0007
01000	1	0.1	0.0066 ← minimum
10001	2	0.1	0.0007
11111	3	0.7	0.0006

From the table, we see that the most likely transmitted codeword is still 01000.